A two-dimensional model of a directional microphone: Calculation of the normal force and moment on the diaphragm

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It has been shown that the parasitoid fly *Ormia Ochracea* exhibits exceptional sound localization ability achieved through the mechanical coupling of its eardrums [R. N. Miles *et al.*, J. Acoust. Soc. Am. **98**, 3059–3070 (1995)]. Based on this biological system a new directional microphone has been designed, having as a basic element a special diaphragm undergoing a rocking motion. This paper considers a 2D model of the microphone in which the diaphragm is considered as a 2D plate having slits on the sides. The slits lead to a backing volume limited by an infinite rigid wall parallel to the diaphragm in its neutral position. The reflection and diffraction of an incoming plane wave by this system are studied to determine the resultant force and resultant moment of pressure upon the diaphragm. The results show that such a microphone will be driven better in the case of narrow slits and deep cavities. [DOI: 10.1121/1.2149838]

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I. INTRODUCTION

The analysis of the auditory system of the parasitoid fly *Ormia Ochracea* revealed a remarkable ability to detect the direction of the incoming sound despite the very small distance between auditory organs.[1] It was determined the fly has a special structure of the auditory system consisting of two closely spaced eardrums with a semirigid bridge connecting them. The mechanical connection between the ears causes them to move in opposite directions in response to the difference in pressure on their exterior surfaces.

Inspired by this biological system and taking advantage of modern MEMS technology Miles *et al.*,[2,3] proposed a new directional microphone integrated on a very small area. The device consists of a polysilicon diaphragm and a backplate to enable capacitive sensing of the diaphragm's motion. The diaphragm is designed to respond like a rigid plate that rocks about a central hinge. Pressure gradients on its exterior result in a net moment about the hinge and cause it to rotate. This rotation is similar to the out-of-phase motions observed in the acoustic response of the fly's ears.

In this paper we consider a simplified 2D model of the acoustic forces on this directional microphone for obtaining information concerning the parameters necessary for design purposes. In Sec. II the geometry of the model is presented, along with the corresponding PDE and the boundary conditions. Thus, the line $A'A$ in Fig. 1(b), is a plane section of the diaphragm in Fig. 1(a), undergoing a rocking motion around the axis Oy. The half-infinite lines $D'B'$ and BD are immobile parts and the segment $E'E$ corresponds to the microphones' backplate. The working domain consists of the upper half-plane $\mathcal{D}^+\equiv\{z>0\}$, the strip $\mathcal{D}^-\equiv\{-h<z<0\}$ and the connecting slits $\mathcal{S}\equiv\{z=0;a<|x|<b\}$. Also, due to the plane-parallel geometry of the domain it is possible to reduce the boundary-value problem (BVP) to a 2D PDE even in the case of a general (plane) incoming wave. After considering this geometrical model of the problem the approach involved is that of general linear acoustics. References for these problems can be found in the classical book by Morse and Ingard[4] (also see Ref. 5). Also, most of the traditional and modern results are collected in the excellent book by Mechel *et al.*[6] The treatment in this paper is mostly analytical. However, for obtaining some results for the analyzed structure in the end some numerical computations are required.[7]

In Sec. III some representation formulas for pressure in the two domains $\mathcal{D}^+$, $\mathcal{D}^-$ are obtained. Next, a Fourier transform with respect to the x-variable was considered and the boundary conditions on the hard surfaces and the condition at infinity were applied. For the acoustical domain of frequencies that we are interested in, the solution consists of a propagating mode and an infinite number of evanescent modes. By imposing the connecting conditions along the slits $\mathcal{S}$ the results of the basic equation of the problem (Sec. IV) can be written as an integral equation. A uniqueness theorem is proven to the solution to this integral equation. Afterwards, the solution is decomposed into an odd and an even part, each of them satisfying a different integral equation. The main part of the kernels of these equations are separated and the regular parts are written in a form suitable to numerical computation. In Sec. V the two integral equations (corresponding to the odd and even parts of the solution) are reduced to two infinite systems of linear equations. This is achieved by using the spectral relationships for the two operators corresponding to the main parts of the kernels. Since the assumed form for the solution takes into consideration the proper behavior of the solution at the points $\pm a$ and $\pm b$, namely square-root singularities, the resulting infinite systems of linear equations have good convergence properties. Formulas are given for computing the resultant force and the resultant moment acting upon the diaphragm in terms of solutions of the infinite systems.

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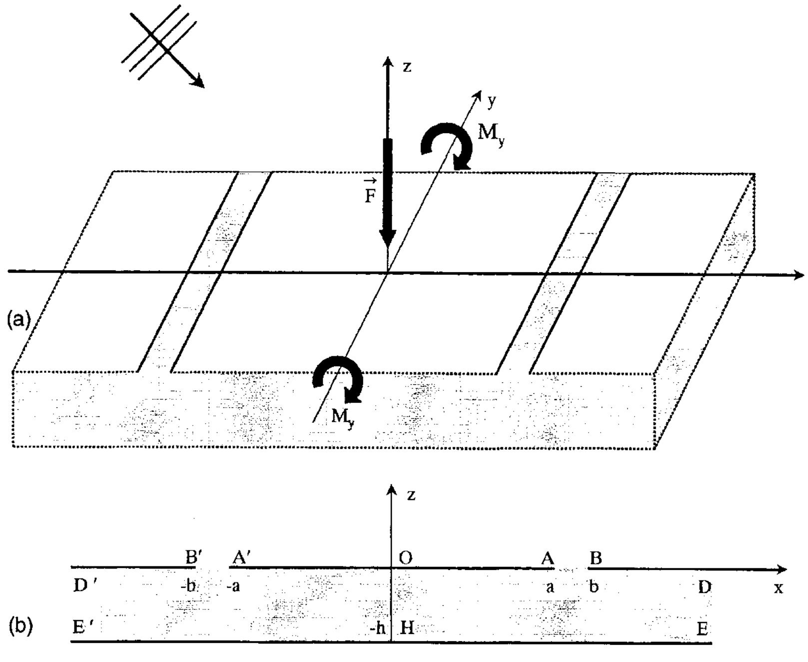

FIG. 1. (a) A sketch of the directional microphone. (b) The 2D model of the directional microphone.

A numerical analysis of the infinite linear systems is given in Sec. VI. It is based on some properties of elliptical functions, a Gauss-Legendre integration formula, and repeated use of the discrete cosine Fourier transform (DCFT). Finally, Sec. VII contains some numerical results. The graph in Fig. 4 gives the total moment of pressure as a function of the slit's width and the microphone's depth. In Figs. 5 and 6 is plotted the force's amplitude and phase delay as a function of the same parameters for a particular value of frequency. The dependence of the moment and force amplitude with frequency is plotted in Figs. 7 and 8.

The conclusion is that a directional microphone built on the ideas in Refs. 1 and 2 is driven better for very small width of the slits and quite deep back-chambers as compared with the diaphragms' width.

II. FORMULATION OF THE PROBLEM

A. The geometry of the model

In order to study the influence of reflection and diffraction of pressure waves by the edges of the diaphragm, on the diaphragm resultant force, and resultant moment we consider the model in Fig. 1. Thus, we assume a plane-parallel geometry in the direction of the Oy-axis. The segment AA' corresponds to the microphone diaphragm, AB and $B'A'$ are the two slits, and EE' is the bottom wall of the die. The origin of the Cartesian system of coordinates has been chosen at the center of the plane-parallel diaphragm and Oz axis on the upward normal direction to the diaphragm plane. We denote also by $\mathcal{D}^+$ the upper domain (the half-plane $z>0$) and by $\mathcal{D}^-$ the strip $-h<z<0$ in the lower half plane. The domain we have to study the motion of the acoustic waves is $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^- \cup S$, where $S = B'A' \cup AB$.

B. The PDE of the problem

In the case of a harmonic motion with respect to time (of ω-angular velocity) we write the perturbation of the pressure p' as

$$p' = p_\omega(x,y,z)e^{-i\omega t}.$$

In this case the scalar wave equation for the pressure becomes

$$\frac{\partial^2 p_\omega}{\partial x^2} + \frac{\partial^2 p_\omega}{\partial y^2} + \frac{\partial^2 p_\omega}{\partial z^2} + \frac{\omega^2}{c_0^2} p_\omega = 0, \tag{1}$$

which is the well known Helmholtz's equation. Here, c_0 is the unperturbed isentropic velocity of the sound.

C. Boundary conditions and condition at infinity

All of the walls are considered to be hard surfaces. Consequently, the normal velocity along the walls will be zero and the following, Neumann-type boundary condition, valid along all of these surfaces, is obtained

$$\frac{\partial p_\omega}{\partial n} = 0, \tag{2}$$

n designates the direction of the normal to the surface.

Concerning the condition at infinity, the system is assumed to be under the action of an incoming plane wave described by the incident-wave direction given by angles θ_i, φ_i of Fig. 2. Thus we have

$$p^i_\omega = \frac{1}{2} \exp\left\{ i\frac{\omega}{c_0}[x \sin \theta_i \cos \varphi_i + y \sin \theta_i \sin \varphi_i - z \cos \theta_i] \right\}. \tag{3}$$

In the case that the domain is $\mathcal{D}^+$ and the boundary is the whole hard plane $z=0$, it can be checked directly that the

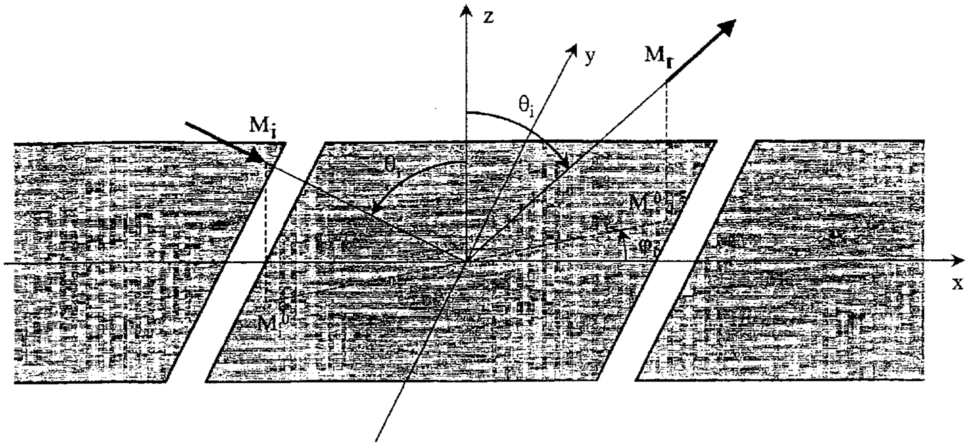

FIG. 2. The incoming and the reflected plane waves.

solution of the problem can be written by adding to p^i_ω, the reflected wave in the form

$$p^0_\omega = \frac{1}{2}\exp\left\{ i\frac{\omega}{c_0}[x \sin\theta_i \cos\varphi_i + y \sin\theta_i \sin\varphi_i - z\cos\theta_i]\right\} + \frac{1}{2}\exp\left\{ i\frac{\omega}{c_0}[x \sin\theta_i \cos\varphi_i + y \sin\theta_i \sin\varphi_i + z\cos\theta_i]\right\}. \tag{4}$$

This formula will be taken as the expression of the solution at infinity (for $z \to \infty$).

In addition, the Sommerfeld Radiation Condition is imposed, so that all of the other propagating perturbations describe outgoing waves.

D. The reduced (2D) PDE

Due to the special geometry of the problem and special form of the condition at infinity, the unknown function $p_\omega(x,y,z)$ shall be written in the form[4]

$$p_\omega(x,y,z) = p(x,z)\exp\left\{ i\frac{\omega}{c_0} y \sin\theta_i \sin\varphi_i\right\}. \tag{5}$$

The new unknown function $p(x,z)$ satisfies the 2D Helmholtz equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + k_0^2 p = 0, \tag{6}$$

where the reduced wave number k_0 has the expression

$$k_0^2 \equiv \frac{\omega_0^2}{c_0^2} = \frac{\omega^2}{c_0^2}(1 - \sin^2\theta_i \sin^2\varphi_i). \tag{7}$$

The solution p^0_ω can also be written as

$$p^0(x,z) = \frac{1}{2}\exp\{ik_0[x\sin\theta_0 - z\cos\theta_0]\} + \frac{1}{2}\exp\{ik_0[x\sin\theta_0 + z\cos\theta_0]\}, \tag{8}$$

where

$$k_0 \sin\theta_0 = \frac{\omega}{c_0}\sin\theta_i \cos\varphi_i,$$

$$k_0 \cos\theta_0 = \frac{\omega}{c_0}\cos\theta_i. \tag{9}$$

III. REPRESENTATION FORMULAS

In order to obtain some representation formulas for the function $p(x,z)$ we write

$$p(x,z) = p^0(x,z) + p^+(x,z), \quad \text{in } \mathcal{D}^+,$$

$$p(x,z) = p^-(x,z), \quad \text{in } \mathcal{D}^-. \tag{10}$$

Both functions $p^+(x,z)$, $p^-(x,z)$ are solutions of the 2D Helmholtz equation (1), satisfying the homogeneous Neumann condition $\partial p/\partial z = 0$ along the walls; on the two slits we can write

$$p^0(x,0) + p^+(x,0) = p^-(x,0), \quad \text{for } x \in S \equiv (-b,-a) \cup (a,b),$$

$$\partial p^+(x,0)/\partial z = \partial p^-(x,0)/\partial z \equiv f(x), \quad \text{for } x \in (-\infty,+\infty), \tag{11}$$

where $f(x)$ vanishes outside the slits and is an unknown function for $x \in S \equiv (-b,-a) \cup (a,b)$. In fact, as all the representation formulas will involve this function, it will be the main unknown function of the problem. This way, the unknown scattered pressures over the open slits are affected directly only by the field arriving from the other parts of the compound diffractor.

A. The solution of the 2D Helmholtz equation in $\mathcal{D}^+$

For determining an expression for the function $p^+(x,z)$ we consider a Fourier transform with respect to x,

$$P^+(\alpha,z) = \int_{-\infty}^{+\infty} p^+(x,z)e^{-i\alpha x}dx.$$

Equation (1) yields the differential equation

$$\frac{d^2P^+(\alpha,z)}{dz^2} - (\alpha^2 - k_0^2)P^+(\alpha,z) = 0. \tag{12}$$

he solution of this equation, which is vanishing at infinity upward, can be written as

$$P^+(\alpha,z) = -\frac{F(\alpha)}{\sqrt{\alpha^2 - k_0^2}} \exp\{-\sqrt{\alpha^2 - k_0^2}z\}. \tag{13}$$

The Fourier transform of the function $f(x)$ is denoted by $F(\alpha)$. Now the function $p^+(x,z)$ can be written by using the convolution theorem

$$p^+(x,z) = -\int_{-\infty}^{+\infty} f(x')K^+(x - x',z)dx', \tag{14}$$

where

$$K^+(x,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\alpha^2 - k_0^2}} \exp\{-\sqrt{\alpha^2 - k_0^2}z\}e^{i\alpha x}d\alpha$$

$$= \frac{i}{2}H_0^1(k_0\sqrt{x^2 + z^2}). \tag{15}$$

Here $H_0^1 = J_0 + iY_0$ is the Hankel function of the first kind of order zero.

B. The solution of the Helmholtz equation in $\mathcal{D}^-$

The solution of Eq. (12) satisfying the homogeneous Neumann boundary condition along the bottom surface and the second condition (11) along the surface $z=0$ can be written as

$$P^-(\alpha,z) = \frac{F(\alpha)}{\sqrt{\alpha^2 - k_0^2}} \frac{\cosh[\sqrt{\alpha^2 - k_0^2}(z + h)]}{\sinh[\sqrt{\alpha^2 - k_0^2}h]}. \tag{16}$$

Hence, the representation formula in $\mathcal{D}^-$ is obtained

$$p^-(x,z) = -\int_{-\infty}^{+\infty} f(x')K^-(x - x',z)dx', \tag{17}$$

where

$$K^-(x,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\alpha^2 - k_0^2}} \frac{\cosh[\sqrt{\alpha^2 - k_0^2}(z + h)]}{\sinh[\sqrt{\alpha^2 - k_0^2}h]} e^{i\alpha x}d\alpha. \tag{18}$$

This function can also be written as

$$K^-(x,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\alpha^2 - t_0^2}} \frac{\cosh\left[\sqrt{\alpha^2 - t_0^2}\left(1 + \frac{z}{h}\right)\right]}{\sinh[\sqrt{\alpha^2 - t_0^2}]} \times \exp\left\{i\alpha\frac{x}{h}\right\}d\alpha, \tag{19}$$

where

$$t_0 = k_0h = \frac{\omega_0}{c_0}h.$$

The inverse Fourier transform in (19) cannot be performed in finite form. However, the residue theorem can be

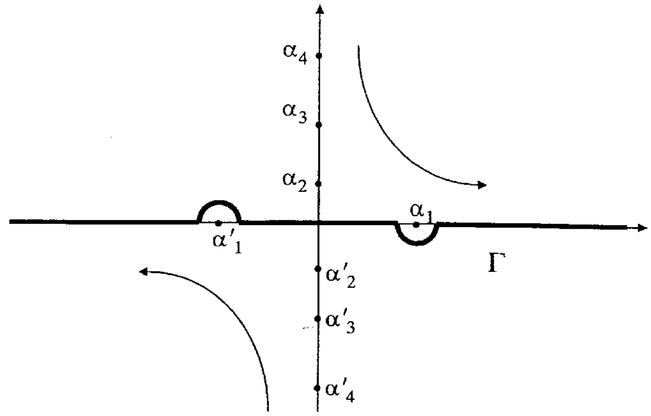

FIG. 3. The integration path Γ.

applied in order to evaluate this integral. There are in this case some real poles which describe undamped waves, corresponding to resonance frequencies, and also some imaginary poles which give evanescent modes. The Sommerfeld condition requires that only the real positive poles which are providing outgoing waves have to be considered. Hence the integration contour will be that drawn in Fig. 3. A simple discussion about the application of the Fourier transform for solving the wave equation can be found in Ref. 5, pp. 293–295.

To be specific, for the range of parameters of this particular problem ($\omega < 2\pi\cdot 20$ kHz) the first pole at the point $\alpha_1 = t_0 (t_0 > 0)$ is real. The next one, $\alpha_2 = \sqrt{t_0^2 - \pi^2}$, is imaginary as well as all the other poles. Consequently the function $K^-(x,y)$ can be written as

$$K^-(x,z) = i\frac{\exp\{ik_0|x|\}}{2t_0} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos[n\pi(1 + z/h)}{\sqrt{n^2\pi^2 - t_0^2}} \times \exp\left\{-\sqrt{n^2\pi^2 - t_0^2}\frac{|x|}{h}\right\}. \tag{20}$$

Now, once the function $f(x)$ is determined, the formulas (14) and (17) can be used for obtaining the pressure field in any point of the domain $\mathcal{D}^-$.

Remark 1: In the case that the working frequency ω is increasing, the poles on the upper imaginary axis are moving toward the real positive semiaxis providing more propagating modes corresponding to different resonance frequencies. Hence, the method developed in this paper can be applied to all (finite) working frequencies if the supplementary propagating modes are taken into consideration.

IV. THE BASIC EQUATION

For obtaining the equation to determine the function $f(x)$, the representation formulas (17) and (20) could be used. Since the formula corresponding to the domain $\mathcal{D}^-$ involves an infinite series it is preferable to work with closed expressions in the Fourier transform plane. Then, the first condition (11) yields the basic equation in the form

$$P_SA(D)f(x) = g(x), \quad x \in S,$$

$$f(x)=0, \quad x\in R-S, \tag{21}$$

where

$$g(x)=\exp\{ik_0 x\sin\theta_0\}.$$

Here P_S is the restriction operator to the reunion of intervals S, and $A(D)$ is a pseudodifferential operator with the symbol

$$A(\alpha)=\frac{1}{\sqrt{\alpha^2-t_0^2}}\frac{\cosh\sqrt{\alpha^2-t_0^2}}{\sinh\sqrt{\alpha^2-t_0^2}}+\frac{1}{\sqrt{\alpha^2-t_0^2}}.$$

The pseudodifferential operator acts on the function $f(x)$ as

$$A(D)f(x)=\frac{1}{2\pi}\int_\Gamma A(\alpha)F(\alpha)\exp\left\{i\alpha\frac{x}{h}\right\}d\alpha.$$

The contour Γ, symmetrical with respect to origin, is shown in Fig. 3.

Thus, for the function $f(x)$ the pseudodifferential equation (21) is obtained. The contour Γ assures that the solution satisfies Sommerfeld's condition at infinity.

Alternatively, the basic equation can be written as the integral equation

$$\int_S f(x')K(x-x')dx'=g(x), \quad x\in S\equiv(-b,-a)\cup(a,b), \tag{22}$$

where the kernel K is

$$K(x)=\frac{1}{2\pi}\int_\Gamma\left[\frac{1}{\sqrt{\alpha^2-t_0^2}}\frac{\cosh\sqrt{\alpha^2-t_0^2}}{\sinh\sqrt{\alpha^2-t_0^2}}+\frac{1}{\sqrt{\alpha^2-t_0^2}}\right]\exp\left\{i\alpha\frac{x}{h}\right\}d\alpha. \tag{23}$$

A. The uniqueness theorem

It shall be proven that the basic equation has at most one solution. Indeed, the homogeneous equation [corresponding to $g(x)\equiv 0$] can be written as

$$\int_S f(x')K(x-x')dx'=0, \quad x\in S.$$

Multiplying by $\overline{f(x)}$ (the overbar denotes the complex conjugate function) and integrating along S there results

$$\int\int K(x-x')f(x')\overline{f(x)}dx'dx=0,$$

the integral being taken over the whole Oxx'-plane. By substituting the expression (23) of the kernel there results

$$\int_\Gamma\left[\frac{1}{\sqrt{\alpha^2-t_0^2}}\frac{\cosh\sqrt{\alpha^2-t_0^2}}{\sinh\sqrt{\alpha^2-t_0^2}}+\frac{1}{\sqrt{\alpha^2-t_0^2}}\right]F(\alpha)\overline{F(\overline{\alpha})}d\alpha=0.$$

Using the residue theorem, in the form valid for the case the contour remains along the real axis, the following expression is obtained:

$$\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\alpha^2-t_0^2}}\frac{\cosh\sqrt{\alpha^2-t_0^2}}{\sinh\sqrt{\alpha^2-t_0^2}}|F(\alpha)|^2d\alpha+\int_{|\alpha|>t_0}\frac{|F(\alpha)|^2}{\sqrt{\alpha^2-t_0^2}}d\alpha+i\left\{\int_{-t_0}^{+t_0}\frac{|F(\alpha)|^2}{\sqrt{t_0^2-\alpha^2}}d\alpha+\pi\frac{|F(-t_0)|^2+|F(t_0)|^2}{2t_0}\right\}=0.$$

The imaginary part in this relation gives

$$F(t_0)=F(-t_0)=0,$$

$$\int_{-t_0}^{+t_0}\frac{|F(\alpha)|^2}{\sqrt{t_0^2-\alpha^2}}d\alpha=0.$$

The last relationship yields $F(\alpha)=0$ almost everywhere in the interval $(-t_0,t_0)$. As the Fourier transform of a summable function over a finite interval is analytic in the whole plane there results $F(\alpha)=0$ all over; its inverse Fourier transform $f(x)$ is also vanishing along the whole real axis. We in fact have proved the uniqueness theorem.

Theorem 2: *Equation (22) has at most a solution in the space of q-summable functions* L_q, $(1<q\leqslant 2)$.

B. Odd and even solutions of the basic equation

The symmetry of the integration intervals with respect to the origin makes it possible to write particular integral equations for the odd and even part of the solution. It is evident from formula (23) that $K(x)=K(-x)$ which is in fact a physical embodiment of reciprocity. Consequently, the kernel is a function of $|x|$ that will be denoted also by $K(|x|)$. We introduce now the odd and even part of the solutions of the basic integral equation by

$$f_o(x)=0.5[f(x)-f(-x)],$$

$$f_e(x)=0.5[f(x)+f(-x)],$$

or, equivalently,

$$f(x)=f_e(x)+f_o(x),$$

$$f(-x)=f_e(x)-f_o(x).$$

Hence, Eq. (22) gives

$$\int_a^b K(|x-x'|)[f_e(x')+f_o(x')]dx'+\int_a^b K(|x+x'|)[f_e(x')-f_o(x')]dx'=g(x), \quad x\in(a,b), \tag{24}$$

$$\int_a^b K(|x+x'|)[f_e(x')+f_o(x')]dx'+\int_a^b K(|x-x'|)[f_e(x')-f_o(x')]dx'=g(-x), \quad x\in(a,b). \tag{25}$$

Now, from the sum and the difference of Eqs. (24) and (25) we obtain

$$\int_a^b K^{(o)}(x,x')f_o(x')=\frac{\pi}{2}g_o(x'), \quad x\in(a,b), \tag{26}$$

$$\int_a^b K^{(e)}(x,x')f_e(x') = \frac{\pi}{2}g_e(x'), \quad x \in (a,b). \tag{27}$$

These are two independent equations for determining the functions $f_o(x)$, $f_e(x)$. We have denoted

$$K^{(o)}(x,x') = \frac{\pi}{2}[K(|x-x'|) - K(|x+x'|)],$$

$$K^{(e)}(x,x') = \frac{\pi}{2}[K(|x-x'|) + K(|x+x'|)],$$

$$g_o(x) \equiv 0.5[g(x) - g(-x)] = i\sin(k_0 x \sin\theta_0),$$

$$g_e(x) \equiv 0.5[g(x) + g(-x)] = \cos(k_0 x \sin\theta_0).$$

C. Analysis of the of integral equation kernels

An analysis of the kernels of the integral equations shall now be produced. Thus, the main (singular) part of the kernel, which determines the behavior (singularity) of the solution at the intervals ends, will be separated. Also, the kernel will be written in a form directly amenable to efficient numerical methods.

Thus,

$$K^{+}(x) \equiv \frac{1}{2\pi}\int_\Gamma \frac{e^{ikx}}{\sqrt{k^2-k_0^2}}dk = -\frac{1}{2}Y_0(k_0|x|) + \frac{i}{2}J_0(k_0|x|),$$

$$K^{-}(x) \equiv \frac{1}{2\pi}\int_\Gamma \frac{\coth\sqrt{t^2-t_0^2}}{\sqrt{t^2-t_0^2}}e^{itx/h}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\coth t_0\sqrt{t^2-1}}{\sqrt{t^2-1}} - \frac{1}{t_0(t^2-1)}\right)e^{ik_0tx}dt + \frac{i}{2t_0}e^{ik_0|x|}.$$

Hence,

$$K^{-}(x) = \frac{1}{\pi}\int_0^1\left\{\frac{1}{t_0(1-t^2)} - \frac{\cot(t_0\sqrt{1-t^2})}{\sqrt{1-t^2}}\right\}\cos(k_0tx)dt + \frac{1}{\pi}\int_1^2\left\{\frac{\coth(t_0\sqrt{t^2-1})}{\sqrt{t^2-1}} - \frac{1}{t_0(t^2-1)} - \frac{1}{\sqrt{t^2-1}}\right\}$$
$$\times\cos(k_0tx)dt + \frac{1}{\pi}\int_2^\infty\left\{\frac{\coth(t_0\sqrt{t^2-1})}{\sqrt{t^2-1}} - \frac{1}{\sqrt{t^2-1}}\right\}$$
$$\times\cos(k_0tx)dt + \frac{1}{\pi t_0}\int_0^2\frac{\cos(k_0tx)}{t^2-1}dt + i\frac{\cos(k_0|x|)}{2t_0} - \frac{1}{2}Y_0(k_0|x|),$$

the last integral is being considered as a Cauchy principal-value integral. In obtaining this formula we have used also the integral

$$\int_0^\infty \frac{\cos(k_0tx)}{t^2-1}dt = -\frac{\pi}{2}\sin(k_0|x|),$$

which can be found in Ref. 8.

We have

$$K(x) = K^{+}(x) + K^{-}(x),$$

and hence, by some changes of variables

$$K(x) = \frac{1}{\pi}\int_0^{\pi/2}\left\{\frac{1}{t_0\sin u} - \cot(t_0\sin u)\right\}\cos(k_0x\cos u)du + \frac{1}{\pi}\int_0^d\left\{\coth(t_0\sinh u) - \frac{1}{t_0\sinh u} - 1\right\}$$
$$\times\cos(k_0x\cosh u)du + \frac{1}{\pi}\int_d^\infty\{\coth(t_0\sinh u) - 1\}$$
$$\times\cos(k_0x\cosh u)du + \frac{1}{\pi t_0}\int_0^2\frac{\cos(k_0xt)}{t^2-1}dt + i\frac{\cos(k_0|x|)}{2t_0} - Y_0(k_0|x|) + \frac{i}{2}J_0(k_0|x|),$$

where $d=\cosh^{-1}(2)$.

Finally the kernel of Eqs. (26) and (27) can be written as

$$K^{(o)}(x,x') = -\ln\frac{|x-x'|}{x+x'} + \frac{1}{t_0}\int_0^2\frac{\sin(k_0xt)\sin(k_0x't)}{t^2-1}dt + \frac{i\pi}{4}\{J_0(k_0|x-x'|) - J_0(k_0|x+x'|)\}$$
$$+\frac{i\pi}{2t_0}\sin(k_0x)\sin(k_0x') - \frac{\pi}{2}\left\{Y_0(k_0|x-x'|) - Y_0(k_0|x+x'|) - \frac{2}{\pi}\ln\frac{|x-x'|}{x+x'}\right\}$$
$$+\int_0^{\pi/2}\left\{\frac{1}{t_0\sin u} - \cot(t_0\sin u)\right\}\sin(k_0x\cos u)\sin(k_0x'\cos u)du$$
$$+\int_0^d\left\{\coth(t_0\sinh u) - \frac{1}{t_0\sinh u} - 1\right\}\sin(k_0x\cosh u)\sin(k_0x'\cosh u)du$$
$$+\int_d^\infty\{\coth(t_0\sinh u) - 1\}\sin(k_0x\cosh u)\sin(k_0x'\cosh u)du,$$

$$K^{(e)}(x,x') = -\ln|x^2 - x'^2| + \frac{1}{t_0}\int_0^2 \frac{\cos(k_0 xt)\cos(k_0 x't)}{t^2-1}dt + \frac{i\pi}{4}\{J_0(k_0|x-x'|) + J_0(k_0|x+x'|)\}$$
$$+ \frac{i\pi}{2t_0}\cos(k_0 x)\cos(k_0 x') - \frac{\pi}{2}\left\{Y_0(k_0|x-x'|) + Y_0(k_0|x+x'|) - \frac{2}{\pi}\ln|x^2-x'^2|\right\}$$
$$+ \int_0^{\pi/2}\left\{\frac{1}{t_0 \sin u} - \cot(t_0 \sin u)\right\}\cos(k_0 x \cos u)\cos(k_0 x' \cos u)du$$
$$+ \int_0^d \left\{\coth(t_0 \sinh u) - \frac{1}{t_0 \sinh u} - 1\right\}\cos(k_0 x \cosh u)\cos(k_0 x' \cosh u)du$$
$$+ \int_d^\infty \{\coth(t_0 \sinh u) - 1\}\cos(k_0 x \cosh u)\cos(k_0 x' \cosh u)du.$$

The first term on the rhs proves that the integral equation has a logarithmic singularity. The first integral is a Cauchy principal-value integral and all the other terms are regular, or regular integrals.

Finally, Eqs. (26) and (27) will be written in the form

$$-\int_a^b f_o(x')\ln\frac{|x-x'|}{x+x'}dx' + \int_a^b f_o(x')\tilde{K}^{(o)}(x,x')dx' = \frac{\pi}{2}g_o(x),$$
$$x \in (a,b), \tag{28}$$

$$-\int_a^b f_e(x')\ln|x-x'^2|dx' + \int_a^b f_o(x')\tilde{K}^{(e)}(x,x')dx' = \frac{\pi}{2}g_e(x),$$
$$x \in (a,b), \tag{29}$$

separating the main (singular) terms,

$$\tilde{K}^{(o)}(x,x') = K^{(o)}(x,x') + \ln\frac{|x-x'|}{x+x'},$$

$$\tilde{K}^{(e)}(x,x') = K^{(e)}(x,x') + \ln|x^2 - x'^2|.$$

V. REDUCTION OF THE INTEGRAL EQUATIONS TO INFINITE SYSTEMS OF LINEAR EQUATIONS

A. The spectral relationships for logarithmic operators and the spaces $L^{1/2}_{2o}(a,b), L^{1/2}_{2e}(a,b)$

For approaching the integral equations (28) and (29) the spectral relationships that invert the main (singular) part of the kernels of the integral equations will be used. For the "odd" case this spectral relationship was given by Aleksandrov *et al.* in Ref. 9,

$$-\int_a^b \frac{T_n(X_o')}{\sqrt{(b^2-x'^2)(x'^2-a^2)}}\ln\frac{|x-x'|}{x+x'}dx' = \mu_n^{(o)}T_n(X_o),$$
$$n \geq 0, \tag{30}$$

where

$$X_o = \cos\left\{\frac{\pi}{K'(c)}F\left(\arcsin\sqrt{\frac{b^2-x^2}{b^2-a^2}},c'\right)\right\}$$

$$\mu_0^{(o)} = \frac{\pi}{b}K(c), \quad \mu_n^{(o)} = \frac{K'(c)}{nb}\tanh\frac{n\pi K(c)}{K'(c)}, \quad n \geq 1,$$

$$c = \frac{a}{b}, \quad c' = \sqrt{1-c^2}, \quad K' = K'(c) = K(c'). \tag{31}$$

$T_n(X_o) = \cos(n \arccos X_o)$ denotes the Chebyshev polynomials of the first kind. F is the elliptic integral of the first kind and by $K(c)$ denotes the complete elliptical integral of the first kind.

Relation (30) yields the formula

$$-\int_a^b\int_a^b \frac{T_n(X_o')T_m(X_o)}{\sqrt{(b^2-x'^2)(x'^2-a^2)}\sqrt{(b^2-x^2)(x^2-a^2)}}$$
$$\times \ln\frac{|x-x'|}{x+x'}dxdx' = \mu_n^{(o)}\int_a^b \frac{T_n(X_o)T_m(X_o)}{\sqrt{(b^2-x^2)(x^2-a^2)}}dx.$$

Due to the symmetry of the first term in this expression with respect to x and x' we can write

$$\mu_n^{(o)}\int_a^b \frac{T_n(X_o)T_m(X_o)}{\sqrt{(b^2-x^2)(x^2-a^2)}}dx = \mu_m^{(o)}\int_a^b \frac{T_n(X_o)T_m(X_o)}{\sqrt{(b^2-x^2)(x^2-a^2)}}dx.$$

As $\mu_n^{(o)} \neq \mu_m^{(o)}$ there results the orthogonality relationship

$$\int_a^b \frac{T_n(X_o)T_m(X_o)}{\sqrt{(b^2-x^2)(x^2-a^2)}}dx = \begin{cases} 0 & \text{for } n \neq m \\ K'(c)/(2b), & \text{for } n = m \neq 0 \\ K'(c)/b, & \text{for } n = m = 0. \end{cases}$$

Inspired by this formula the inner product

$$\langle f(x), g(x)\rangle_o = \int_a^b \frac{f(X_o)\overline{g(X_o)}}{\sqrt{(b^2-x^2)(x^2-a^2)}}dx$$

is defined and, correspondingly, the norm

$$\|f(x)\|_o^2 = \int_a^b \frac{|f(X_o)|^2}{\sqrt{(b^2-x^2)(x^2-a^2)}}dx.$$

We denote by $\mathbf{L}_{2o}^{1/2}(a,b)$ the completion of the space of continuous functions on (a,b) in this norm. It is obvious that $\mathbf{L}_{2o}^{1/2}(a,b)$ is a Hilbert space. In this space the set of functions $T_n(X_o)$ is an orthogonal complete system.

A similar analysis can be made for the main (singular) part of the "even" equation. In this case the spectral relationship is

$$-\int_a^b \frac{2x' T_n(X_e')}{\sqrt{(b^2-x'^2)(x'^2-a^2)}} \ln|x^2-x'^2| dx' = \mu_n^{(e)} T_n(X_e),$$

$$n \geqslant 0,$$

where

$$X_e = \frac{2}{b^2-a^2}\left[x^2 - \frac{b^2+a^2}{2}\right],$$

$$\mu_0^{(e)} = \pi \ln \frac{4}{b^2-a^2}, \quad \mu_n^{(e)} = \frac{\pi}{n}, \quad n \geqslant 1.$$

The orthogonality relationship

$$\int_a^b \frac{2xT_n(X_e)T_m(X_e)}{\sqrt{(b^2-x^2)(x^2-a^2)}} dx = \begin{cases} 0 & \text{for } n \neq m \\ \pi/2, & \text{for } n = m \neq 0 \\ \pi, & \text{for } n = m = 0 \end{cases}$$

can be proven and also the "even" inner product

$$\langle f(x), g(x)\rangle_e = \int_a^b \frac{2xf(X_e)\overline{g(X_e)}}{\sqrt{(b^2-x^2)(x^2-a^2)}} dx$$

may be defined along with the corresponding norm

$$\|f(x)\|_e^2 = \int_a^b \frac{2x|f(X_e)|^2}{\sqrt{(b^2-x^2)(x^2-a^2)}} dx.$$

We denote by $\mathbf{L}_{2e}^{1/2}(a,b)$ the completion of the space of continuous functions on $[a,b]$ in the $\mathbf{L}_{2e}^{1/2}(a,b)$ norm. The set of polynomials $T_n(X_e)$ is an orthogonal complete system in this Hilbert space.

B. The infinite system of linear equations equivalent to the "odd" integral equation

According to the theory developed in previous section the solution of the integral equation (28) can be written in the form

$$f_o(x) = \frac{\sum_{n=0}^{\infty} A_n^{(o)} T_n(X_o)}{\sqrt{(b^2-x^2)(x^2-a^2)}}, \tag{32}$$

where the coefficients $A_n^{(o)}$ have to be determined. Equation (28) becomes

$$\sum_{n=0}^{\infty} \mu_n^{(o)} A_n^{(o)} T_n(X_o) + \sum_{n=0}^{\infty} A_n^{(o)} \int_a^b \frac{T_n(X_o')\widetilde{K}^{(o)}(x,x')dx'}{\sqrt{(b^2-x'^2)(x'^2-a^2)}}$$
$$= \frac{\pi}{2} g_o(x). \tag{33}$$

In order to obtain a system of linear equations for unknown coefficients A_n consider the inner product with $T_m(X_o)$. [The spectral postmultiplication of the integral equation with the function $T_m(X_o)$ is somewhat arbitrary since one cannot hope to achieve a spectral diagonal (perfect separability). Rather one hopes that the Chebyshev polynomial function in question will strengthen the diagonal of the resulting spectral system of equations.] The relationship (33) becomes

$$\delta_m^{(o)2} \mu_m^{(o)} A_m^{(o)} + \sum_{n=0}^{\infty} \widetilde{K}_{mn}^{(o)} A_n^{(o)} = g_m^{(o)}, m = 0,1,2,\ldots, \tag{34}$$

where

$$\delta_m^{(o)2} = \begin{cases} K'(c)/(2b), & \text{for } m \neq 0 \\ K'(c)/b & \text{for } m = 0, \end{cases}$$

$$\widetilde{K}_{mn}^{(o)} = \int_a^b \int_a^b \frac{T_m(X_o)T_n(X_o')\widetilde{K}^{(o)}(x,x')dxdx'}{\sqrt{(b^2-x^2)(x^2-a^2)}\sqrt{(b^2-x'^2)(x'^2-a^2)}}, \tag{35}$$

$$g_m^{(o)} = \frac{\pi}{2}\int_a^b \frac{T_m(X_o)g_o(x)dx}{\sqrt{(b^2-x^2)(x^2-a^2)}}. \tag{36}$$

The way the infinite linear system was obtained (34) shows that the system is equivalent to the integral equation (28). Since the linear system comes directly from the integral equation the proof of uniqueness also applies to the linear system. That is the infinite linear system always has an unique solution.

Remark 3: The conversion of the relationship (33) into a linear system (34) is equivalent to a Galerkin formulation. Alternatively, a collocation method can be used by imposing Eq. (33) to be satisfied at a special set of points. In this case the Galerkin procedure is preferred due to its connection with the above defined inner products.

C. The infinite system of linear equations equivalent to the "even" integral equation

The solution of the integral equation (29) shall be written in the form

$$f_e(x) = \frac{2x\sum_{n=0}^{\infty} A_n^{(e)} T_n(X_e)}{\sqrt{(b^2-x^2)(x^2-a^2)}}, \tag{37}$$

where the coefficients $A_n^{(e)}$ have to be determined from the integral equation. We obtain

$$\sum_{n=0}^{\infty} \mu_n^{(e)} A_n^{(e)} T_n(X_e) + \sum_{n=0}^{\infty} A_n^{(e)} \int_a^b \frac{2x' T_n(X_e')\widetilde{K}(x,x')dx'}{\sqrt{(b^2-x'^2)(x'^2-a^2)}}$$
$$= \frac{\pi}{2} g_e(x). \tag{38}$$

The "even" inner product with $T_m(X_e)$ yields the infinite system of linear equations

$$\delta_m^{(e)2} \mu_m^{(e)} A_m^{(e)} + \sum_{n=0}^{\infty} \widetilde{K}_{mn}^{(e)} A_n^{(e)} = g_m^{(e)}, \quad m = 0,1,2,\ldots, \tag{39}$$

where

$$\delta_m^{(e)2} = \begin{cases} \pi/2, & \text{for } m \neq 0 \\ \pi, & \text{for } m = 0. \end{cases}$$

$$\widetilde{K}_{mn}^{(e)} = \int_a^b \int_a^b \frac{4xx' T_m(X_e) T_n(X_e') \widetilde{K}^{(e)}(x,x') dx dx'}{\sqrt{(b^2-x^2)(x^2-a^2)}\sqrt{(b^2-x'^2)(x'^2-a^2)}}, \tag{40}$$

$$g_m^{(e)} = \frac{\pi}{2} \int_a^b \frac{2x T_m(X_e) g_e(x) dx}{\sqrt{(b^2-x^2)(x^2-a^2)}}. \tag{41}$$

D. The forces due to fluid pressure upon the diaphragm

In order to obtain the resultant force and the resultant moment of pressure upon the diaphragm consider the relationships

$$p_\omega(x,-0) = \int_S f(x') K^-(|x-x'|) dx',$$

$$p_\omega(x,+0) = p^0(x,0) - \int_S f(x') K^+(|x-x'|) dx',$$

giving the pressure on two faces of the diaphragm in terms of the function $f(x)$. Hence

$$p_\omega(x,-0) - p_\omega(x,+0) = -p^0(x,0) + \int_S f(x') K(|x-x'|) dx'.$$

By introducing the odd and even parts of the function f we can write this relationship in the form

$$\begin{aligned} p_\omega(x,-0) - p_\omega(x,+0) = & -p^0(x,0) \\ & + \frac{2}{\pi}\int_a^b f_o(x') K_o(x,x') dx' \\ & + \frac{2}{\pi}\int_a^b f_e(x') K_e(x,x') dx'. \end{aligned}$$

Hence,

$$\begin{aligned} F &\equiv \int_{-a}^{+a} [p_\omega(x,-0) - p_\omega(x,+0)] dx \\ &= -\int_{-a}^{+a} p^0(x,0) dx + \frac{2}{\pi}\int_{-a}^{+a} dx \int_a^b f_e(x') K_e(x,x') dx', \end{aligned}$$

$$\begin{aligned} M &\equiv \int_{-a}^{+a} x[p_\omega(x,-0) - p_\omega(x,+0)] dx \\ &= -\int_{-a}^{+a} x p^0(x,0) dx + \frac{2}{\pi}\int_{-a}^{+a} x dx \int_a^b f_o(x') K_o(x,x') dx'. \end{aligned}$$

By using the expressions (32) and (37) of the two functions f_o, f_e we obtain

$$F = -2\frac{\sin(ak_0 \sin\theta_0)}{k_0 \sin\theta_0} + \sum_{m=0}^{\infty} A_m^{(e)} F_m, \tag{42}$$

$$M = 2ai\frac{\cos(ak_0 \sin\theta_0)}{k_0 \sin\theta_0} - 2i\frac{\sin(ak_0 \sin\theta_0)}{(k_0 \sin\theta_0)^2} + \sum_{m=0}^{\infty} A_m^{(o)} M_m, \tag{43}$$

where

$$F_n = \frac{4}{\pi}\int_0^a dx \int_a^b \frac{2x' T_n(X_e') K^{(e)}(x,x') dx'}{\sqrt{(b^2-x'^2)(x'^2-a^2)}}, \tag{44}$$

$$M_n = \frac{4}{\pi}\int_0^a x dx \int_a^b \frac{T_n(X_o') K^{(o)}(x,x') dx'}{\sqrt{(b^2-x'^2)(x'^2-a^2)}}. \tag{45}$$

Remark 4: We note that due to 2D model considered in this paper the pressure does not depend upon the y-variable. Correspondingly, the total normal force and moment on the diaphragm can be obtained by multiplying the quantities F and M [given by formulas (42) and (43)] by L_y, the diaphragm's width.

VI. NUMERICAL ANALYSIS OF THE LINEAR SYSTEMS

For both the "odd" and "even" cases an infinite system of linear equations for solving the problem was obtained. However, the coefficients of these systems cannot be evaluated analytically, hence they must be computed by numerical methods.

A. The "odd" case

To begin consider the problem of computation of generalized Fourier coefficients of a given continuous function with respect to the orthogonal bases $\{T_m(X_o)\}$ in the Hilbert space $\mathbf{L}_{2o}^{1/2}(a,b)$. For example,

$$g_m^{(o)} = \frac{\pi}{2}\langle g_o(x), T_m(X_o)\rangle.$$

Let $h(x)$ be a smooth function defined on the interval (a,b). Then,

$$\langle h(x), T_m(X_o)\rangle_o = \int_a^b \frac{h(x) T_m(X_o) dx}{\sqrt{(b^2-x^2)(x^2-a^2)}}.$$

By inverting the relationship (31) the following expression for x can be written:

$$x = \sqrt{b^2 - (b^2-a^2)\mathrm{sn}^2\left[\frac{K'}{\pi}\arccos(X_o)\right]}, \tag{46}$$

where sn is the Jacobian elliptic function and,

$$X_o = \cos\theta. \tag{47}$$

By means of the changes of variables (46) and (47) there results

$$\begin{aligned} &\int_a^b \frac{h(x) T_m(X_o) dx}{\sqrt{(b^2-x^2)(x^2-a^2)}} \\ &= \frac{K'}{\pi b}\int_0^\pi h\left(\sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta\right)}\right)\cos(m\theta) d\theta. \end{aligned} \tag{48}$$

By using the relationship[8]

$$\mathrm{sn}(u+2K',c') = -\mathrm{sn}(u,c'),$$

we can write

$$\mathrm{sn}^2\left[\frac{K'}{\pi}(\theta+2\pi),c'\right] = \mathrm{sn}^2\left(\frac{K'}{\pi}\theta+2K',c'\right) = \mathrm{sn}^2\left(\frac{K'}{\pi}\theta,c'\right).$$

Hence, the function

$$\tilde{h} \equiv h\left(\sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta\right)}\right)$$

is a smooth, even, 2π-periodic function. The generalized Fourier coefficient can be written as

$$h_n^{(o)} \equiv \langle h(x),T_m(X_o)\rangle_o = \frac{K'}{2b}\tilde{h}_m,$$

where

$$\tilde{h}_m = \frac{1}{\pi}\int_0^{2\pi} h\left(\sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta\right)}\right)\cos(m\theta)d\theta,$$

$$m=0,1,\ldots$$

are the (cosine) Fourier coefficients of the smooth function $\tilde{h}(\theta)$. They can be well approximated by means of the discrete cosine Fourier transform and can be efficiently computed by using a DCT algorithm based on FFT.[10]

For determining the coefficients $\tilde{K}_{mn}^{(o)}$, consider the expression

$$h_{mn}^{(o)} = \langle\langle h(x,x'),T_n(X_o')\rangle_o,T_m(X_o)\rangle_o = \int_a^b\int_a^b \frac{T_m(X_o)T_n(X_o')h(x,x')dxdx'}{\sqrt{(b^2-x^2)(x^2-a^2)}\sqrt{(b^2-x'^2)(x'^2-a^2)}}.$$

By using the change of variables

$$\begin{cases} x = \sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta\right)} \\ x' = \sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta'\right)}, \end{cases}$$

there results

$$h_{mn}^{(o)} = \frac{K'^2}{4\pi^2b^2}\int_0^{2\pi}\int_0^{2\pi} h\left(\sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta\right)},\right.$$
$$\left.\sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta'\right)}\right)\cdot\cos(m\theta)\cos(n\theta')d\theta d\theta'.$$

Thus, in the case the function $h(x,y)$ is given by its analytical form, the coefficients $h_{mn}^{(o)}$ can be computed by means of a 2D discrete cosine (Fourier) transform.

In order to evaluate the contribution to $\tilde{K}_{mn}^{(o)}$ coefficients of the integral terms in the expression of the kernel $K^{(o)}\times(x,x')$ we introduce the functions

$$S_n^o(k_0t) = \langle\sin(k_0tx),T_n(X_o)\rangle_o$$

which can be computed as

$$S_n^o(k_0t) = \frac{K'}{2\pi b}\int_0^{2\pi}\sin\left[k_0t\sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta\right)}\right]\cos(n\theta)d\theta$$

by using a DCT algorithm. Now due to the special form of the integrals a parallel algorithm for computing simultaneously all of the integrals corresponding to all values $0\leqslant n\leqslant N$, $0\leqslant m\leqslant n$ by certain quadrature formulas may be employed. Consider, as an example, the Cauchy principal-value integral

$$I_1^o \equiv \int_a^b\int_a^b dxdx'\int_0^2 \frac{\sin(k_0tx)T_m(X_o)}{\sqrt{(b^2-x^2)(x^2-a^2)}} \times\frac{\sin(k_0tx')T_n(X_o')}{\sqrt{(b^2-x'^2)(x'^2-a^2)}}\frac{dt}{t^2-1}.$$

We can write

$$I_1^o = \int_0^2 \frac{S_n^o(k_0t)S_m^o(k_0t)}{t^2-1}dt = \sum_{j=1}^{2p}\frac{w_j}{t_j^2-1}S_n^o(k_0t_j)S_m^o(k_0tj),$$

w_j,t_j being the weights and, respectively, the nodes of an even $(2p)$ Gauss-Legendre quadrature formula.

By the same method the moment coefficients M_n may be obtained

$$M_n = \frac{2K'}{\pi^2b}\int_0^a xdx\int_0^{2\pi}K^{(o)} \times\left(x,\sqrt{b^2-(b^2-a^2)\mathrm{sn}^2\left(\frac{K'}{\pi}\theta\right)}\right)\cos(n\theta)d\theta.$$

For the explicit terms in the kernel $K^{(o)}$ the calculation is straightforward. For the terms containing integrals take as an example the same integral as before. Thus,

$$IM_1 \equiv \frac{4}{\pi}\int_0^a xdx\int_a^b \frac{T_n(X_o')dx'}{\sqrt{(b^2-x'^2)(x'^2-a^2)}} \times\int_0^2 \frac{\sin(k_0xt)\sin(k_0x't)}{t^2-1}dt = \frac{4}{\pi} \times\int_0^2\left(\frac{\sin(k_0at)}{k_0^2t^2} - a\frac{\cos(k_0at)}{k_0t}\right)\frac{S_n^o(k_0t)}{t^2-1}dt.$$

The resulting integral can also be computed by means of an even Gauss-Legendre quadrature formula.

B. The "even" case

A similar analysis can be performed for the "even" problem. In this case only the final formulas shall be given. Thus, for a smooth function $h(x)$ along the interval (a,b)

$$\langle h(x),T_m(X_e)\rangle_e=\int_a^b \frac{2xh(x)T_m(X_e)dx}{\sqrt{(b^2-x^2)(x^2-a^2)}}$$
$$=\frac{\pi}{2}\frac{1}{\pi}\int_0^{2\pi} h\left(\sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}\cos\theta}\right)$$
$$\times\cos(m\theta)d\theta$$

is obtained. Hence,

$$g_m^{(e)}=\left(\frac{\pi}{2}\right)^2\hat{h}_m,$$

where $\hat{h}_m$ are the cosines Fourier coefficients of the even, 2π-periodic function

$$\hat{h}(\theta)=h\left(\sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}\cos\theta}\right).$$

Also, for the coefficients

$$h_{mn}^{(e)}=\langle\langle h(x,x'),T_n(X_e')\rangle_e,T_m(X_e)\rangle_e$$
$$=\int_a^b\int_a^b \frac{4xx'T_m(X_e)T_n(X_e')h(x,x')dxdx'}{\sqrt{(b^2-x^2)(x^2-a^2)}\sqrt{(b^2-x'^2)(x'^2-a^2)}}$$

there results

$$h_{mn}^{(e)}=\frac{1}{4}\int_0^{2\pi}\int_0^{2\pi} h\left(\sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}\cos\theta},\right.$$
$$\left.\sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}\cos\theta}\right)\cdot\cos(m\theta)\cos(n\theta')d\theta d\theta',$$

a formula which can be used in the case an analytical expression is given for the function $h(x,x')$ by means of a 2D cosine Fourier transform.

The terms containing integrals are defined as

$$S_n^e(k_0t)=\langle\sin(k_0tx),T_n(X_e)\rangle_e$$
$$=\frac{1}{2}\int_0^{2\pi}\sin\left[k_0t\sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}\cos\theta}\right]\cos(n\theta)d\theta.$$

Hence,

$$I_1^e\equiv\int_a^b\int_a^b dxdx'\int_0^2\frac{4xx'\sin(k_0tx)T_m(X_e)}{\sqrt{(b^2-x^2)(x^2-a^2)}}$$
$$\times\frac{\sin(k_0tx')T_n(X_e')}{\sqrt{(b^2-x'^2)(x'^2-a^2)}}\frac{dt}{t^2-1}$$
$$=\int_0^2\frac{S_n^e(k_0t)S_m^e(k_0t)}{t^2-1}dt=\sum_{j=1}^{2p}\frac{w_j}{t_j^2-1}S_n^e(k_0t_j)S_m^e(k_0t_j).$$

Finally

$$F_n=\frac{4}{\pi}\int_0^a dx\int_0^{2\pi}K^{(o)}$$
$$\times\left(x,\sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}\cos\theta}\right)\cos(n\theta)d\theta$$

for the explicit part of the kernel and

$$IF_1\equiv\frac{4}{\pi}\int_0^a dx\int_a^b\frac{2x'T_n(X_e')dx'}{\sqrt{(b^2-x'^2)(x'^2-a^2)}}$$
$$\times\int_0^2\frac{\sin(k_0xt)\sin(k_0x't)}{t^2-1}dt=\frac{2}{\pi}$$
$$\times\int_0^2\left(\frac{1-\cos(k_0at)}{k_0t}\right)\frac{S_n^e(k_0t)}{t^2-1}dt$$

for the part containing integrals which cannot be obtained in closed form. The last integrals can be computed (for all the values of n) by using a vectorized form of an even Gauss-Legendre quadrature formula.

VII. NUMERICAL RESULTS

The parameters in the present problem are: the angles θ_i,φ_i giving the incident-wave direction, a, the diaphragm's half length, b-a the slit's width, the depth h, and frequency $f=\omega/2\pi$. We consider $a=1$ such that all the lengths are normalized with respect to half-diaphragm length.

For determining the pressure on the diaphragm we have to solve firstly the systems (34) and (39) for the coefficients $A_n^{(o)}$ and $A_n^{(e)}$. As the representation formulas (32) and (37) assure the proper behavior of the pressure at the points a and b the infinite systems have good convergence properties, such that only a few terms have to be retained in the infinite systems. Once the parameters $A_n^{(o)}$ and $A_n^{(e)}$ are determined, the moment M and the force F are provided by formulas (43) and (42), respectively.

The numerical values $M(\theta_i,\varphi_i)$ obtained by solving the problem satisfy the relationship

$$M(\theta_i,\varphi_i)=M\cos\varphi_i\sin\theta_i,$$

where

$$M=M(\pi/2,0).$$

We denote by Mo and Fo the moment and the force corresponding to the incoming and reflected wave when $b=a$ (the solution of the problem without slits) and have plotted in Fig. 4 the value of the ratio M/Mo for certain values of $h>0$ and $b>1$. Also, in Fig. 5 is plotted the amplitude of

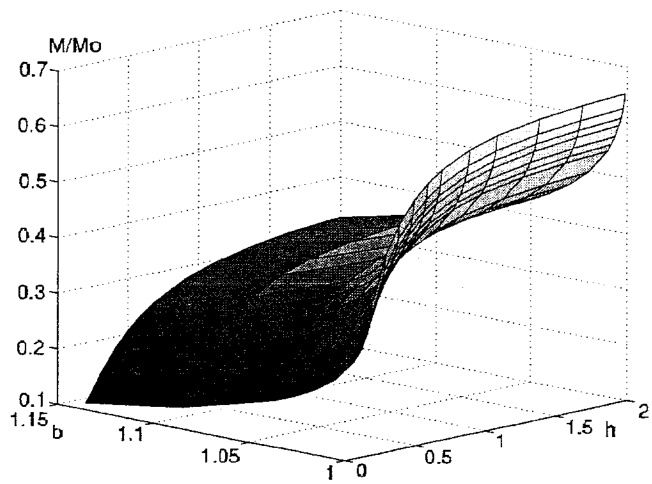

FIG. 4. Variation of the moment ratio M/M_o with b and h at a frequency of 10 kHz.

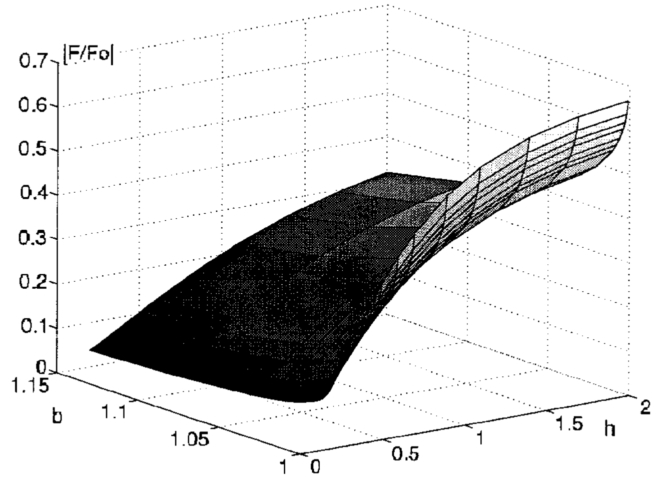

FIG. 5. Variation of the modulus of force ratio F/F_o with b and h at a frequency of 10 kHz.

the ratio F/Fo. It is to be noticed that in contrast to the moment, the force F has a phase delay plotted in Fig. 6.

Figures 4 and 5 point out that the interesting case from the design point of view, characterized by larger values of moment and force, is that of values of b closer to 1 and larger values of h. All the calculations were carried out for the frequency value $f = 10$ kHz.

Some physical explanations of these results can be provided by analyzing the diffraction of a plane wave by a grating. This problem, in the case of a simplified one-mode approximation, has an explicit analytical solution[11,12] and a plot of the transmission coefficient is given in Ref. 13, Fig. 2(b). It is clear that the transmission coefficient is significantly lower than unity only in the case of very narrow slits. For wider slits, an important part of the incoming plane wave is passing through slits in the lower half-plane, equalizing the pressure on the two faces of the diaphragm. As the diaphragm is driven mainly by the pressure difference on the two faces of the diaphragm (or by the net moment due to the pressure difference) it is clear that the device will work better in the case of very narrow slits. In the case of the finite depth of the backing cavity, the lower wall of the backchamber will give a reflection of the waves which have penetrated the slits giving an additional increase of the pressure on the lower face of the diaphragm and decreasing the pressure difference between the two faces.

Next, we investigate the dependence of moment and force upon frequency. These functions are plotted in Figs. 7 and 8. The normalized force amplitude $|F/Fo|$ is increasing nearly linearly with frequency while the ratio M/Mo is practically independent of frequency.

The analysis performed in this paper considers a simplified model for the directional microphone. There are many aspects of the analysis and design to consider in the development of the device. This paper focusses only on the applied moments and forces due to sound. Finally, we assumed in this work that the diaphragm is a rigid plate. The elasticity

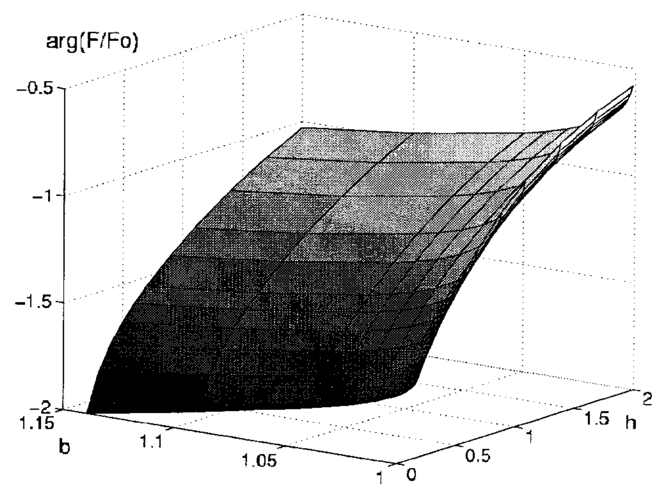

FIG. 6. Variation of the phase of force ratio F/F_o with b and h at a frequency of 10 kHz.

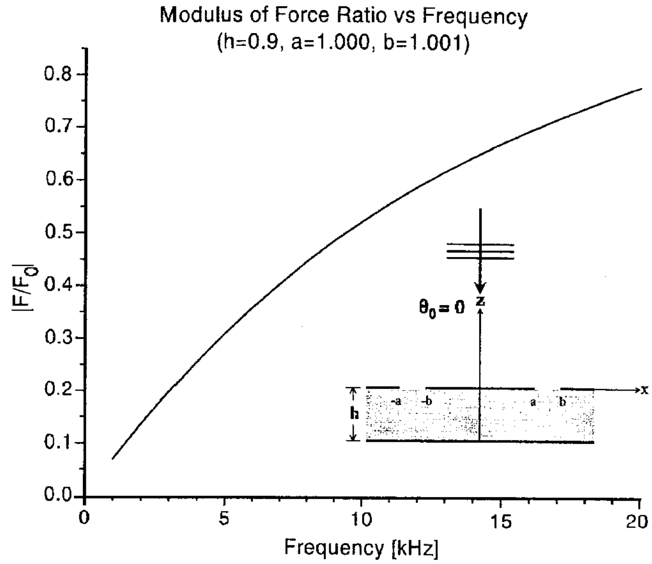

FIG. 7. Increase of the modulus of the force ratio with frequency ($a = 1.0$ mm, $b = 1.001$ mm, $h = 0.9$ mm, $\theta_0 = 0$).

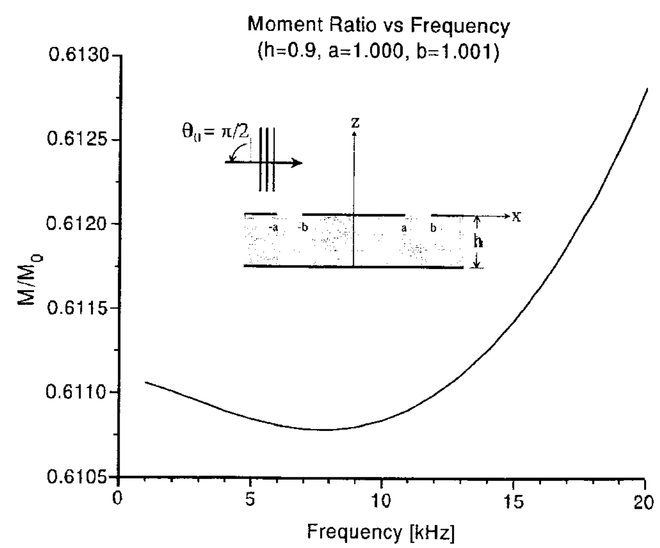

FIG. 8. Variation of the resultant moment of pressure with frequency ($a = 1.0$ mm, $b = 1.001$ mm, $h = 0.9$ mm, $\theta_0 = 90°$).

of the real diaphragms will give supplementary problems which can be addressed only in a more completed computational model.

ACKNOWLEDGMENT

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